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Simultaneous Mass, Damping, and Stiffness Updating for Dynamic Systems

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This paper presents an efficient and systematic approach to simultaneously update the mass, damping, and stiffness matrices of linear dynamic systems, given few (say two) measured complex vibration modes (complex eigenvalues and eigenvectors). The method is termed the *cross-model cross-mode* method because it involves solving a set of linear simultaneous equations in which each equation is formulated based on the product terms from two same/different modes associated with the mathematical and experimental models, respectively. Two numerical examples are demonstrated: a 4-degree-of-freedom mass–spring–damper system and a 30-degree-of-freedom finite element model for a cantilever beam. The numerical updating by the cross-model cross-mode method is excellent for all system matrices when the measured modes are spatially complete and noise free. The cross-model cross-mode method, together with the Guyan reduction scheme, also performs reasonably well under a spatial incompleteness situation.

Nomenclature

Scales

$C^{\dagger}_{ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{C} \mathbf{\Phi}'_j$
$C^{\dagger}_{n,ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{C}_n \mathbf{\Phi}$
$C^{\dagger}_{n,m}$	=	same as $C^{\dagger}_{n,ij}$ with new index m to replace ij
c_n	=	nth damping coefficient of the mass–spring–damper model
$f^{\dagger}_m$	=	see Eq. (21)
$K^{\dagger}_{ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{K} \mathbf{\Phi}'_j$
$K^{\dagger}_{n,ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{K}_n \mathbf{\Phi}'_j$
$K^{\dagger}_{n,m}$	=	same as $K^{\dagger}_{n,ij}$ with new index m to replace ij
k_n	=	nth stiffness coefficient of the mass–spring–damper model
$M^{\dagger}_{ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{M} \mathbf{\Phi}'_j$
$M^{\dagger}_{n,ij}$	=	$(\mathbf{\Phi}_i)^T \mathbf{M}_n \mathbf{\Phi}'_j$
$M^{\dagger}_{n,m}$	=	same as $M^{\dagger}_{n,ij}$ with new index m to replace ij
m_n	=	nth mass coefficient of the mass–spring–damper model
N_C	=	number of correction coefficients for the damping matrix
N_i	=	number of modes from the baseline model
N_j	=	number of modes from the updated model
N_K	=	number of correction coefficients for the stiffness matrix
N_M	=	number of correction coefficients for the mass matrix
N_m	=	$N_i \times N_j$
α_n	=	nth element of $\boldsymbol{\alpha}$
β_n	=	nth element of $\boldsymbol{\beta}$
γ_n	=	nth element of $\boldsymbol{\gamma}$
λ_j, λ'_j, λ''_j	=	jth eigenvalue of baseline, updated, and true models, respectively
ξ_r	=	damping ratio of the rth mode of the dynamic system
ω_r	=	natural frequency of the rth mode of the dynamic system

Vectors and Matrices

$\mathbf{C}$, $\mathbf{C}'$, $\mathbf{C}''$	=	damping matrix of the baseline, updated, and true models, respectively
$\mathbf{C}_n$	=	preselected damping submatrix of the baseline model
$\mathbf{C}^{\dagger}$	=	$N_m \times N_C$ complex matrix
$\mathbf{c}_n$	=	element damping matrix of the nth element of the analytical model
$\mathbf{F}^{\dagger}$	=	real column vector of size $2N_m$, Eq. (25)
$\mathbf{f}^{\dagger}$	=	complex column vector of size N_m
$\mathbf{G}^{\dagger}$	=	see Eq. (24)
$\mathbf{G}^{\sharp}$	=	pseudoinverse of $\mathbf{G}^{\dagger}$, Eq. (27)
$\mathbf{K}$, $\mathbf{K}'$ $\mathbf{K}''$	=	stiffness matrix of the baseline, updated, and true models, respectively
$\mathbf{K}_n$	=	preselected stiffness submatrix of the baseline model
$\mathbf{K}^{\dagger}$	=	$N_m \times N_K$ complex matrix
$\mathbf{k}_n$	=	element stiffness matrix of the nth element of the mass–spring–damper model
$\mathbf{M}$, $\mathbf{M}'$, $\mathbf{M}''$	=	mass matrix of the baseline, updated, and true models, respectively
$\mathbf{M}_n$	=	preselected mass submatrix of the baseline model
$\mathbf{M}^{\dagger}$	=	$N_m \times N_M$ complex matrix
$\mathbf{x}$, $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$	=	displacement, velocity, and acceleration vectors, respectively
$\boldsymbol{\alpha}$	=	vector of correction coefficients for the stiffness matrix
$\boldsymbol{\beta}$	=	vector of correction coefficients for the mass matrix
Γ	=	$[\boldsymbol{\alpha}\ \ \boldsymbol{\beta}\ \ \boldsymbol{\gamma}]^T$
$\boldsymbol{\gamma}$	=	vector of correction coefficients for the damping matrix
$\mathbf{\Phi}_j$, $\mathbf{\Phi}'_j$, $\mathbf{\Phi}''_j$	=	jth mode shape of the baseline, updated, and true models, respectively

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Superscripts

T	=	superscript used as transpose operator
†	=	superscript used for "cross" terms between baseline and updated models
$*$	=	superscript used for complex conjugates

Introduction

DISAGREEMENTS often exist between analytical models and experimentally obtained data. Model updating, defined as the process of correcting the numerical values of individual parameters in a mathematical model using data obtained from an associated experimental model such that the updated model more correctly describes the dynamic properties of the subject structure, has become a common method to improve the correlation between finite element models and measured data [1–3]. A number of approaches to the problem exist, based on the type of parameters that are updated and the measured data that are used. This study belongs to the category of updating the complete structural matrices, including mass, damping, and stiffness matrices, from modal data.

Most papers in the area of finite element model updating (FEMU) addressed undamped systems. Traditionally, the modal-based FEMU for undamped systems can be classified into two major groups: *direct matrix methods* [4–6] and *indirect physical property adjustment methods* [1]. The first of these two groups is generally of noniterative methods, all of which were based on computing changes made directly to the mass and stiffness matrices. Such changes may have succeeded in generating modified models which had properties close to those measured in the tests, but these resulting models become abstract "representation" models and cannot be interpreted in a physical way. The second group is of those methods that are in many ways more acceptable in that the parameters which they adjust are much closer to physically realizable quantities. Methods in this second group seek to find correction factors for each individual finite element or for each design parameter relating to each finite element and are generally viewed as the main hope for updating technology even though they require a much greater computation effort. They are all iterative, in contrast to the direct formulas of the earlier methods. Taking a completely different approach from the traditional methods, Hu et al. [7] recently developed the *cross-model cross-mode* (CMCM) model updating method for the simultaneous updating of the stiffness and mass matrices. The method was so named because it involves solving a set of linear simultaneous equations for the physically meaningful correction factors, in which each equation is formulated based on the product terms from two same/different modes associated with the mathematical and experimental models, respectively. The CMCM method is a noniterative method and therefore very cost effective in computational time. It also has the advantage of preserving the initial model configuration and physical connectivity of the updated model. All the FEMU methods mentioned previously used real-valued mode shapes and natural frequencies.

Applying FEMU to damped systems was first conducted by Friswell et al. [8], who extended the traditional direct methods to estimating both the damping and stiffness matrices of a damaged cantilever beam while assuming that its mass matrix was known. Their algorithm has the drawback that it does not guarantee the connectivity of the original finite element model. Kuo et al. [9] extended the direct method to a more general problem where the analytical mass, damping, and stiffness matrices were all allowed to be updated. Model updating for damped systems also appeared in the literature under the mathematical term *quadratic inverse eigenvalue problems* (QIEP). There are many articles in the area of QIEP. Particularly, Starek and Inman [10] studied the QIEP associated with nonproportional underdamped systems. The latest progress in solving QIEP has been detailed in a recent book by Chu and Golub [11].

A common practice in the industry to measure the modes of a real dynamic system has been the experimental modal analysis (EMA). The EMA data are usually characterized by a small number of modes, which are likely to be spatially incomplete and complex valued [1]. Although complex modes occur in real dynamic systems for a variety of reasons, vibration modes of conventional (i.e., nonrotating) linear structures can be complex only if the damping is distributed in a nonproportional way. In real structures, nonproportional damping can arise readily because the majority of the damping is found to be concentrated at the joints between components of a structural assembly that does not result in a proportional distribution in damping. This paper further develops the CMCM method to render it suitable to damped systems, where the damping matrix can be either proportional or nonproportional. The precise objective is the reconstruction of mass, damping, and stiffness matrices using few measured complex modes, while maintaining the physical connectivity of the mathematical model. Two particular structural models—a 4-degree-of-freedom (DOF) mass–spring–damper system and a 30-DOF cantilever beam structure—are to be chosen for the numerical examples, where the measured modal information will be synthesized from using a finite element model that is similar to the analytical model, but with different sets of system coefficients. In the numerical studies, scenarios with both spatially complete and incomplete modes will be investigated.

Preliminaries: Eigenanalysis of Linear Damped Dynamic Systems

Consider the homogeneous equations of motion for a linear structure

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are mass, damping, and stiffness matrices, respectively; and $\mathbf{x}$, $\dot{\mathbf{x}}$, and $\ddot{\mathbf{x}}$ are displacement, velocity, and acceleration vectors, respectively. Matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are restricted to be real and symmetric. Let the solution to Eq. (1) have the form:

$$\mathbf{x} = \mathbf{\Phi} e^{\lambda t} \tag{2}$$

Substituting Eq. (2) and its time derivatives into Eq. (1) yields

$$(\lambda^2 \mathbf{M} + \lambda \mathbf{C} + \mathbf{K})\mathbf{\Phi} = \mathbf{0} \tag{3}$$

The solution of Eq. (3) constitutes a complex eigenproblem. When the structure is with N degrees of freedom, there are $2N$ eigenvalues occurring in complex conjugate pairs, as a result of the fact that all the coefficients in the matrices are real and thus any eigenvalues (or roots) must either be real or occur in complex conjugate pairs. The corresponding eigenvectors to these eigenvalues also occur as complex conjugates. Denote the eigensolution as λ_r, λ_r^*, $\mathbf{\Phi}_r$, and $\mathbf{\Phi}_r^*$, $r = 1, \ldots, N$. In vibration analyses, it is customary to express each eigenvalue λ_r in the form:

$$\lambda_r = \omega_r(-\xi_r + i\sqrt{1 - \xi_r^2}) \tag{4}$$

where $i = \sqrt{-1}$, ω_r is the *natural frequency*, and ξ_r is the *damping ratio* for that mode. The significance of complex eigenvectors is that the mode shapes are complex. In effect, a complex mode is one where each part of the structure has not only its own amplitude of vibration but also its own phase. As a result, each part of a structure which is vibrating in a complex mode will reach its own maximum deflection at a different instant in the vibration cycle to that of its neighbors which all have different phases.

Often numerical libraries only provide first order eigenvalue solvers, thus a transformation from Eq. (3) to its first order form is needed. For keeping symmetric square matrices for the generalized eigenvalue analysis, a common $2N$ state space representation of Eq. (1) can be expressed as

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{x}} \\ \ddot{\mathbf{x}} \end{Bmatrix} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{Bmatrix} \tag{5}$$

Solving the generalized eigenproblem of Eq. (5) yields the eigenvalue λ_r and its corresponding eigenvector in the form as

$$\lambda_r\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{0} \end{bmatrix}\begin{Bmatrix} \mathbf{\Phi}_r \\ \lambda_r\mathbf{\Phi}_r \end{Bmatrix} = \begin{bmatrix} -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}\begin{Bmatrix} \mathbf{\Phi}_r \\ \lambda_r\mathbf{\Phi}_r \end{Bmatrix} \tag{6}$$

One thus can immediately obtain the eigenvector $\mathbf{\Phi}_r$ associated with Eq. (3) from extracting the first half of the eigenvector shown in Eq. (6).

Cross-Model Cross-Mode Method Using Complex Modes

Throughout this paper, to distinguish symbols associated with the models before and after updating, without a superscript is used for the original (or baseline) model, and the superscript "′" is used for the updated model. For instance, $\mathbf{M}'$ and $\mathbf{M}$ represent the mass matrix of the updated and baseline models, respectively.

First, the stiffness, damping, and mass matrices of the baseline model, denoted as $\mathbf{K}$, $\mathbf{C}$, and $\mathbf{M}$, respectively, have been initially modeled. In the proposed cross-model cross-mode approach, the stiffness matrix $\mathbf{K}'$ of the updated model is a modification of $\mathbf{K}$ via

$$\mathbf{K}' = \mathbf{K} + \sum_{n=1}^{N_K} \alpha_n \mathbf{K}_n \tag{7}$$

where an individual $\mathbf{K}_n$ is a preselected stiffness submatrix of the baseline model; α_n are unknown stiffness correction factors to be determined; and N_K is the number of stiffness correction terms for the stiffness matrix. Often $\mathbf{K}_n$ could be chosen to be the stiffness matrix of each element, and then N_K the number of elements. Likewise, one writes the corresponding expression for the mass matrix $\mathbf{M}'$ and viscous damping matrix $\mathbf{C}'$, respectively, as

$$\mathbf{M}' = \mathbf{M} + \sum_{n=1}^{N_M} \beta_n \mathbf{M}_n \tag{8}$$

and

$$\mathbf{C}' = \mathbf{C} + \sum_{n=1}^{N_C} \gamma_n \mathbf{C}_n \tag{9}$$

where the individual $\mathbf{C}_n$ and $\mathbf{M}_n$ are preselected submatrices of the baseline model; β_n and γ_n are correction coefficients to be determined; and N_M and N_C are the number of mass and damping correction coefficients, respectively.

Denoting the jth eigenvalue λ'_j and eigenvector $\mathbf{\Phi}'_j$ associated with $\mathbf{K}'$, $\mathbf{C}'$, and $\mathbf{M}'$, one writes

$$\mathbf{K}'\mathbf{\Phi}'_j + \lambda'_j\mathbf{C}'\mathbf{\Phi}'_j + \lambda'^2_j\mathbf{M}'\mathbf{\Phi}'_j = \mathbf{0} \tag{10}$$

In the following development, it is assumed that few λ'_j and $\mathbf{\Phi}'_j$ are known from measurements. Denoting superscript "T" as a transpose operator, and premultiplying Eq. (10) by $(\mathbf{\Phi}_i)^T$, which can be either a real or a complex mode, yields

$$(\mathbf{\Phi}_i)^T\mathbf{K}'\mathbf{\Phi}'_j + \lambda'_j(\mathbf{\Phi}_i)^T\mathbf{C}'\mathbf{\Phi}'_j + \lambda'^2_j(\mathbf{\Phi}_i)^T\mathbf{M}'\mathbf{\Phi}'_j = 0 \tag{11}$$

Substituting Eqs. (7–9) into the above equation yields

$$K^\dagger_{ij} + \sum_{n=1}^{N_K}\alpha_n K^\dagger_{n,ij} + \lambda'_j\left(C^\dagger_{ij} + \sum_{n=1}^{N_C}\gamma_n C^\dagger_{n,ij}\right) + \lambda'^2_j\left(M^\dagger_{ij} + \sum_{n=1}^{N_M}\beta_n M^\dagger_{n,ij}\right) = 0 \tag{12}$$

where

$$M^\dagger_{ij} = (\mathbf{\Phi}_i)^T\mathbf{M}\mathbf{\Phi}'_j \tag{13}$$

$$C^\dagger_{ij} = (\mathbf{\Phi}_i)^T\mathbf{C}\mathbf{\Phi}'_j \tag{14}$$

$$K^\dagger_{ij} = (\mathbf{\Phi}_i)^T\mathbf{K}\mathbf{\Phi}'_j \tag{15}$$

$$K^\dagger_{n,ij} = (\mathbf{\Phi}_i)^T\mathbf{K}_n\mathbf{\Phi}'_j \tag{16}$$

$$M^\dagger_{n,ij} = (\mathbf{\Phi}_i)^T\mathbf{M}_n\mathbf{\Phi}'_j \tag{17}$$

and

$$C^\dagger_{n,ij} = (\mathbf{\Phi}_i)^T\mathbf{C}_n\mathbf{\Phi}'_j \tag{18}$$

For clarity, symbols with superscript "†" throughout this paper are "cross" terms calculated from both baseline and updated models. Using a new index m to replace ij, Eq. (12) becomes

$$K^\dagger_m + \sum_{n=1}^{N_K}\alpha_n K^\dagger_{n,m} + \lambda'_j\left(C^\dagger_m + \sum_{n=1}^{N_C}\gamma_n C^\dagger_{n,m}\right) + \lambda'^2_j\left(M^\dagger_m + \sum_{n=1}^{N_M}\beta_n M^\dagger_{n,m}\right) = 0 \tag{19}$$

Rearranging Eq. (19), one obtains

$$\sum_{n=1}^{N_M}\beta_n(\lambda'^2_j M^\dagger_{n,m}) + \sum_{n=1}^{N_C}\gamma_n(\lambda'_j C^\dagger_{n,m}) + \sum_{n=1}^{N_K}\alpha_n K^\dagger_{n,m} = f^\dagger_m \tag{20}$$

where

$$f^\dagger_m = -(\lambda'^2_j M^\dagger_m + \lambda'_j C^\dagger_m + K^\dagger_m) \tag{21}$$

When N_i and N_j modes are taken from the analytical model and the measured model, respectively, totally $N_m = N_i \times N_j$ complex equations can be formed from Eq. (20). Those equations are named CMCM equations in view of the fact that they are formed by crossing over two models, for two arbitrary modes. Equation (20) can be rewritten in a matrix form:

$$\mathbf{K}^\dagger\boldsymbol{\alpha} + \mathbf{M}^\dagger\boldsymbol{\beta} + \mathbf{C}^\dagger\boldsymbol{\gamma} = \mathbf{f}^\dagger \tag{22}$$

where $\mathbf{K}^\dagger$, $\mathbf{M}^\dagger$, and $\mathbf{C}^\dagger$ are N_m-by-N_K, N_m-by-N_M, and N_m-by-N_C complex matrices, respectively. $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ are real column vectors of size N_K, N_M, and N_C; and $\mathbf{f}^\dagger$ is a complex column vector of size N_m. Denoting Re(z) and Im(z) as the real and imaginary parts of a complex number z, respectively, one can rewrite Eq. (22) as

$$\mathbf{G}^\dagger\Gamma = \mathbf{F}^\dagger \tag{23}$$

where

$$\mathbf{G}^\dagger = \begin{bmatrix} \mathrm{Re}(\mathbf{K}^\dagger) & \mathrm{Re}(\mathbf{M}^\dagger) & \mathrm{Re}(\mathbf{C}^\dagger) \\ \mathrm{Im}(\mathbf{K}^\dagger) & \mathrm{Im}(\mathbf{M}^\dagger) & \mathrm{Im}(\mathbf{C}^\dagger) \end{bmatrix} \tag{24}$$

$$\mathbf{F}^\dagger = \begin{bmatrix} \mathrm{Re}(\mathbf{f}^\dagger) \\ \mathrm{Im}(\mathbf{f}^\dagger) \end{bmatrix} \tag{25}$$

and

$$\Gamma = \begin{Bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{26}$$

Note that there are $2N_m$ real-valued CMCM equations in Eq. (23).

Analytically, one can solve Γ in Eq. (23) by a standard inverse operation, $\Gamma = \mathbf{G}^{\dagger^{-1}}\mathbf{F}^\dagger$, if $\mathbf{G}^\dagger$ is a nonsingular square matrix. For a nonsquare matrix $\mathbf{G}^\dagger$, where the number of equations does not equal the number of unknowns, the equivalent operator is the pseudoinverse. If $\mathbf{G}^\dagger$ has more rows than columns, an overdetermined case where there are more equations than unknowns, that is $2N_m > (N_M + N_C + N_K)$, the pseudoinverse is defined as

$$\mathbf{G}^\sharp = (\mathbf{G}^{\dagger^T}\mathbf{G}^\dagger)^{-1}\mathbf{G}^{\dagger^T} \tag{27}$$

for nonsingular $(\mathbf{G}^{\dagger^T}\mathbf{G}^\dagger)$. The resulting solution, $\Gamma = \mathbf{G}^\sharp\mathbf{F}^\dagger$ is optimal in a least-squares sense.

When all nonzero stiffness, damping, and mass coefficients of the baseline model are allowed to change, this particular situation can be termed a *complete-updating* case, in contrast to a *partial-updating* case where one or more nonzero coefficients of $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are not allowed to vary. If the updated matrices $\mathbf{K}'$, $\mathbf{M}'$, and $\mathbf{C}'$ in Eqs. (7–9) are replaced by $a\mathbf{K}'$, $a\mathbf{M}'$, and $a\mathbf{C}'$, where a is an arbitrary positive constant not equal to 1, it would correspond to a different set of correction factors. Although these two dynamic systems, characterized by $(\mathbf{M}', \mathbf{C}', \mathbf{K}')$ and $(a\mathbf{M}', a\mathbf{C}', a\mathbf{K}')$, respectively, are different in spatial domain, they are identical in the modal space, namely, they possess the same eigenvalues λ'_j and eigenvectors $\mathbf{\Phi}'_j$. Because the CMCM equations in Eq. (22) or Eq. (23) are derived based on the usage of λ'_j and $\mathbf{\Phi}'_j$, the corresponding solutions for the correction factors should apply to both dynamic systems. From the above statements, one concludes that multiple sets of solution for the correction factors must exist for a complete-updating case. In theory, to gain a unique solution for the correction factors, at least an additional constraint equation must be imposed. For instance, a particular mass or stiffness term is predetermined, or the total mass of the system is known.

Numerical Studies

Numerical examples are given to illustrate the procedure of applying the CMCM method and to demonstrate the accuracy of the method for correcting the stiffness, mass, and damping coefficients of an analytical finite element model based on measured complex modes. In this paper, both the analytical model and the measured modal information are generated from using finite element models, with different sets of system coefficients. It is assumed that the analytical model is with a "wrong" set of coefficients, and the goal is to correct those wrong coefficients from the measured modal information which is simulated from the true model with the "right" coefficients. In the following presentation, the term *analytical model* refers to the model with $\mathbf{K}$, $\mathbf{M}$, and $\mathbf{C}$, and the term *true model* refers to the model with $\mathbf{K}''$, $\mathbf{M}''$, and $\mathbf{C}''$. Throughout this paper, symbols with superscript "$''$" always denote quantities associated with the true model. Although the true model and the updated model share the same symbolic expression such as Eqs. (7–9), they might be different numerically. Two particular structural models—a 4-DOF lumped mass–spring–damper system, and a 30-DOF cantilever beam structure—are chosen for the numerical examples. Those structural models were investigated previously by Pilkey [12] and Friswell et al. [8], respectively.

4-DOF Mass–Spring–Damper System

Consider the 4-DOF mass–spring–damper system shown in Fig. 1. This 4-DOF structure was studied by Pilkey [12] who focused on estimating the damping matrix while assuming that mass and stiffness matrices were known. Using this simple example allows the detailed numerical results being presented, and also provides the reader an opportunity to verify the correctness of the CMCM method independently. The mass–spring–damper model also has been investigated by many researchers in the area of inverse eigenvalue problems. It is mathematically equivalent to a shear building model, to a lumped-mass finite element model of a rod in longitudinal vibration, to a set of point masses vibrating transversely on a taut string, and to a finite difference or finite element approximation to a Sturm–Liouville problem [13].

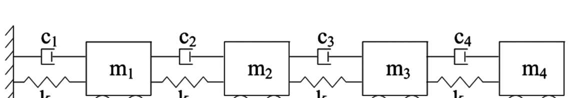

Fig. 1 Sketch of a 4-DOF mass–spring–damper system.

Following the concept of the finite element method, the element stiffness and damping matrices of the nth element that connects the $(n-1)$th and nth masses are given as [14]

$$\mathbf{k}_n = \begin{bmatrix} k_n & -k_n \\ -k_n & k_n \end{bmatrix} \tag{28}$$

and

$$\mathbf{c}_n = \begin{bmatrix} c_n & -c_n \\ -c_n & c_n \end{bmatrix} \tag{29}$$

The system stiffness matrix $\mathbf{K}$ for a 4-DOF mass–spring–damper system can be assembled as

$$\mathbf{K} = \sum_{n=1}^{4}\mathbf{K}_n = \begin{bmatrix} k_1+k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3+k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 \end{bmatrix} \tag{30}$$

where $\mathbf{K}_n$ denotes the corresponding $\mathbf{k}_n$ in the global coordinates, and the system damping matrix $\mathbf{C}$:

$$\mathbf{C} = \sum_{n=1}^{4}\mathbf{C}_n = \begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 \\ 0 & 0 & -c_4 & c_4 \end{bmatrix} \tag{31}$$

where $\mathbf{C}_n$ denotes the corresponding $\mathbf{c}_n$ in the global coordinates. The corresponding system mass matrix $\mathbf{M}$ is written as

$$\mathbf{M} = \sum_{n=1}^{4}\mathbf{M}_n = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_3 & 0 \\ 0 & 0 & 0 & m_4 \end{bmatrix} \tag{32}$$

where $\mathbf{M}_n$ denotes the corresponding m_n in the global coordinates.

Numerical values for the system coefficients of the analytical model are chosen to be identical to those of Pilkey [12]: $m_1 = m_4 = 5$, $m_2 = m_3 = 10$; $k_n = 1$ and $c_n = 0.01$ for $n = 1, \ldots, 4$. The system matrices $\mathbf{K}$, $\mathbf{C}$, and $\mathbf{M}$ of the analytical model are numerically determined from using Eqs. (30–32). Because the eigenvalues and eigenvectors occur in complex conjugate pairs, only one root from each pair is needed. Performing the eigenanalysis based on Eq. (6), one obtains the four eigenvalues as $\lambda_1 = -0.0001 + 0.1256i$, $\lambda_2 = -0.0007 + 0.3864i$, $\lambda_3 = -0.0018 + 0.5922i$, and $\lambda_4 = -0.0024 + 0.6959i$, and the corresponding eigenvectors are

$$\mathbf{\Phi}_1 = \begin{Bmatrix} 1.0000 \\ 1.9211 \\ 2.5392 \\ 2.7566 \end{Bmatrix}, \qquad \mathbf{\Phi}_2 = \begin{Bmatrix} 1.0000 \\ 1.2535 \\ -0.3646 \\ -1.4383 \end{Bmatrix}$$

$$\mathbf{\Phi}_3 = \begin{Bmatrix} 1.0000 \\ 0.2465 \\ -1.3715 \\ 1.8202 \end{Bmatrix}, \qquad \mathbf{\Phi}_4 = \begin{Bmatrix} 1.0000 \\ -0.4211 \\ 0.1969 \\ -0.1386 \end{Bmatrix}$$

Those eigenvectors are obtained to be real vectors because the system damping of the analytical model has been taken to be a proportional damping, noting $\mathbf{C} = 0.01\mathbf{K}$.

The true system stiffness matrix $\mathbf{K}''$ is synthesized by using Eq. (7) with the above $\mathbf{K}$ and assigned quantities $\alpha_1 = 0.4$, $\alpha_2 = 0.1$, $\alpha_3 = -0.2$, and $\alpha_4 = -0.1$. Similarly, the way to produce $\mathbf{M}''$ and $\mathbf{C}''$ is according to Eqs. (8) and (9) with $\beta_1 = -0.1$, $\beta_2 = 0.2$, $\beta_3 = -0.3$, $\beta_4 = 0.35$, $\gamma_1 = -0.1$, $\gamma_2 = 0.3$, $\gamma_3 = -0.2$, and

$\gamma_4 = 0.15$. Sequentially, performing the eigenanalysis associated with $\mathbf{M}''$, $\mathbf{K}''$, and $\mathbf{C}''$, one obtains the four eigenvalues as $\lambda_1'' = -0.0001 + 0.1329i$, $\lambda_2'' = -0.0006 + 0.3408i$, $\lambda_3'' = -0.0020 + 0.5901i$, and $\lambda_4'' = -0.0028 + 0.7798i$, and the corresponding complex eigenvectors:

$$\mathbf{\Phi}_1'' = \begin{Bmatrix} 1.0000 - 0.0000i \\ 2.2004 - 0.0009i \\ 3.2676 - 0.0015i \\ 3.7670 - 0.0019i \end{Bmatrix}, \qquad \mathbf{\Phi}_2'' = \begin{Bmatrix} 1.0000 - 0.0000i \\ 1.7977 - 0.0021i \\ -0.2365 - 0.0010i \\ -1.8311 + 0.0024i \end{Bmatrix}$$

$$\mathbf{\Phi}_3'' = \begin{Bmatrix} 1.0000 - 0.0000i \\ 0.8482 - 0.0039i \\ -3.7909 + 0.0066i \\ 2.3523 - 0.0017i \end{Bmatrix}, \qquad \mathbf{\Phi}_4'' = \begin{Bmatrix} 1.0000 - 0.0000i \\ -0.2149 - 0.0006i \\ 0.0746 + 0.0004i \\ -0.021 - 0.0002i \end{Bmatrix}$$

In the current implementation of the CMCM method, all four modes from the analytical model and the first two complex modes from the true model are employed. Thus, total 16 real-valued CMCM equations can be formed. An additional constraint equation (for scaling purpose) is that the change of the total mass of the system is presumably known. In the present numerical example, one has

$$\sum_{n=1}^{4} \beta_n m_n = 0.25$$

to be consistent with the simulated data. There are 12 unknown correction coefficients to be estimated. With 17 equations and 12 unknowns, one solves it as an overdetermined case. Shown in Fig. 2 is resulting estimates of the stiffness correction coefficients α_n, the mass correction coefficients β_n, and the damping correction coefficients γ_n for $n = 1, \ldots, 4$, plotted against those coefficients originally used to generate $\mathbf{K}''$, $\mathbf{M}''$, and $\mathbf{C}''$. It is found that the correction factors estimated from applying the CMCM method match perfectly with the preset values.

30-DOF Cantilever Beam System

The CMCM method will be tested on a 10-element steel cantilever beam shown in Fig. 3, where each element is modeled as a uniform beam element. The beam is of length 1 m, breadth 25 mm, and thickness 50 mm. Young's modulus is taken as 2.05×10^{11} Pa, and mass density $7860\ \mathrm{kg/m^3}$ (or mass density per unit length $9.825\ \mathrm{Kg/m}$). The cross-section area and the associated moment of

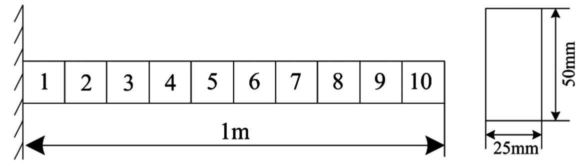

Fig. 3 Sketch of a 10-element cantilever beam.

inertia are $1.25 \times 10^{-3}\ \mathrm{m^2}$ and $2.604 \times 10^{-7}\ \mathrm{m^4}$, respectively. Following the standard formulation for a uniform beam element in a plane, one computes the element stiffness matrix $\mathbf{k}_n$ and element (consistent) mass matrix $\mathbf{m}_n$ of a size 6×6. There are 10 nodal points and each nodal point has 3 DOFs, thus matrices $\mathbf{K}$ and $\mathbf{M}$ both are with a size 30×30. With 10 structural elements, the global stiffness matrix of the analytical model is obtained as

$$\mathbf{K} = \sum_{n=1}^{10} \mathbf{K}_n$$

and that of the true model as

$$\mathbf{K}'' = \sum_{n=1}^{10} (1 + \alpha_n)\mathbf{K}_n$$

where $\mathbf{K}_n$ is the corresponding $\mathbf{k}_n$ in global coordinates. Likewise, for the global mass matrix of the analytical and true models, one has

$$\mathbf{M} = \sum_{n=1}^{10} \mathbf{M}_n$$

and

$$\mathbf{M}'' = \sum_{n=1}^{10} (1 + \beta_n)\mathbf{M}_n$$

respectively, where $\mathbf{M}_n$ is the corresponding $\mathbf{m}_n$ in global coordinates. The analytical model is considered to be an undamped system, that is $\mathbf{C} = \mathbf{0}$, and the global damping matrix of the true model is assumed to have the form

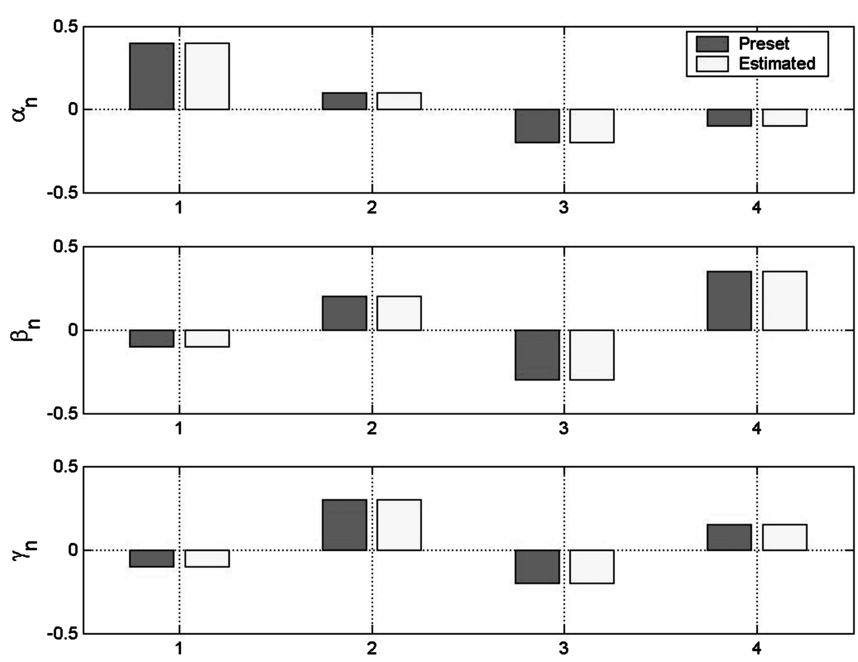

Fig. 2 Comparison of the preset and estimated correction coefficients α_n, β_n, and γ_n for the 4-DOF mass–spring–damper system.

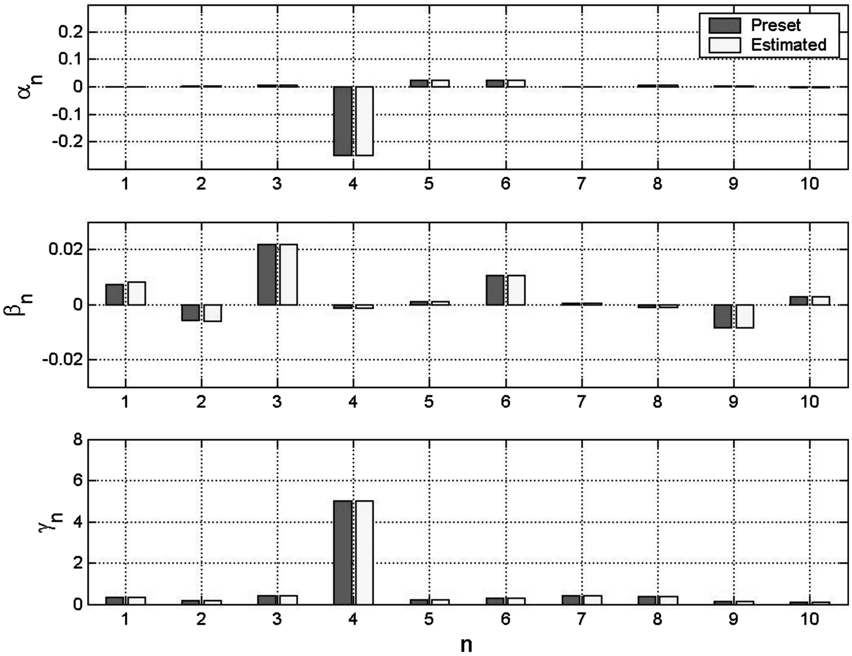

Fig. 4 Comparison of the preset and estimated correction coefficients α_n, β_n, and γ_n for the beam model when two spatially complete modes are used.

$$\mathbf{C}'' = \sum_{n=1}^{10} \gamma_n \mathbf{C}_n$$

where $\mathbf{C}_n$ is considered to have a scaled form of $\mathbf{K}_n$. Precisely, it is assumed $\mathbf{C}_n = 10^{-5}\mathbf{K}_n$. Note that the interpretation of γ_n differs from that of α_n or β_n. The term $\gamma_n \mathbf{C}_n$ is an extra damping term from zero damping, rather than a modification of an existent $\mathbf{C}_n$. Although γ_n must be positive to justify a positive energy dissipation, the value of α_n or β_n must be greater than -1 to justify a positive stiffness or mass term.

Similar to the case study in Friswell et al. [8], the true model is considered to be a damaged cantilever beam. The damage is modeled as a reduction in the stiffness of element 4 by 25% from the analytical model, that is $\alpha_4 = -0.25$. The damping in element 4 is assumed to be 5×10^{-5} times the stiffness of element 4, that is $\gamma_4 = 5$. This damaged model is motivated by the realization that damage often reduces stiffness and adds damping locally. In addition to $\alpha_4 = -0.25$ and $\gamma_4 = 5$, the parameters of the true model are considered to be slightly different from those of the analytical model for other elements as well. Specifically, the true model is produced with the quantities α_n generated by using a Gaussian random number generator with the mean equal to 0 and standard deviation equal to 0.02, and β_n being generated based on mean 0 and standard deviation 0.01. Physically, damping terms must be positive. The quantities for γ_n are generated by taking the absolute value of the Gaussian random numbers based on mean 0 and standard deviation 0.25.

In the present CMCM implementation, 10 bending modes from the analytical model and two complex bending modes from the true model are used, thus 40 real-valued CMCM equations can be formed. Those CMCM equations, together with a scaling constraint equation that the first stiffness term is unchanged, that is, $\alpha_1 = 0$, are sufficient to solve for the 30 correction coefficients. Shown in the top panel of Fig. 4 are the resulting estimations of the stiffness correction coefficients α_n, plotted against the preset coefficients which were used to generate $\mathbf{K}''$. Likewise, the estimated and preset mass correction coefficients β_n and damping correction coefficients γ_n are shown in the middle and bottom panels of Fig. 4, respectively. For providing quantitative detail of those correction coefficients shown in Fig. 4, Table 1 lists all the preset and estimated values as well as their relative errors. At the damaged element, the relative errors of α_4 and γ_4 are 8.73×10^{-7} and -4.05×10^{-7}, respectively, which are practically negligible. The worst updating occurs at β_1, which has a relative error 0.12. Overall, the numerical results indicate that the correction coefficients estimated from applying the CMCM method match nicely with the target values.

In the above calculation, only two complex modes of the damaged beam are used, but all DOFs of the measured mode are assumed known. If only a subset of the DOFs is measured (a spatially incomplete situation), then the analytical model must be reduced or the measured mode shapes must be expanded. When only the translational DOFs in the transverse direction are measured for the first two complex modes and the Guyan reduction scheme [15] is used to reduce the analytical model, implementing the CMCM

Table 1 The preset correction coefficients (denoted by α_n, β_n, and γ_n) and their estimations (denoted by $\hat{\alpha}_n$, $\hat{\beta}_n$, and $\hat{\gamma}_n$) for the beam model when two spatially complete modes are used

n	α_n	$\hat{\alpha}_n$	$\frac{\hat{\alpha}_n-\alpha_n}{\alpha_n}$	β_n	$\hat{\beta}_n$	$\frac{\hat{\beta}_n-\beta_n}{\beta_n}$	γ_n	$\hat{\gamma}_n$	$\frac{\hat{\gamma}_n-\gamma_n}{\gamma_n}$
1	0	−1.22e − 17	——	7.26e − 03	8.13e − 03	1.20e − 01	3.34e − 01	3.34e − 01	−2.06e − 07
2	2.51e − 03	2.51e − 03	−7.28e − 06	−5.88e − 03	−6.00e − 03	1.93e − 02	1.79e − 01	1.79e − 01	2.32e − 07
3	5.75e − 03	5.75e − 03	1.38e − 05	2.18e − 02	2.19e − 02	1.18e − 03	4.06e − 01	4.06e − 01	−4.72e − 07
4	−2.50e − 01	−2.50e − 01	8.73e − 07	−1.36e − 03	−1.36e − 03	−3.04e − 05	5.00e + 00	5.00e + 00	−4.05e − 07
5	2.38e − 02	2.38e − 02	2.65e − 05	1.14e − 03	1.13e − 03	−9.52e − 03	2.15e − 01	2.15e − 01	1.66e − 06
6	2.38e − 02	2.38e − 02	−4.32e − 05	1.07e − 02	1.07e − 02	1.53e − 03	3.14e − 01	3.14e − 01	−8.98e − 07
7	−7.53e − 04	−7.51e − 04	−2.61e − 03	5.93e − 04	5.75e − 04	−3.07e − 02	3.98e − 01	3.98e − 01	1.08e − 06
8	6.55e − 03	6.54e − 03	−7.75e − 04	−9.56e − 04	−9.42e − 04	−1.55e − 02	3.60e − 01	3.60e − 01	−2.34e − 06
9	3.49e − 03	3.51e − 03	4.09e − 03	−8.32e − 03	−8.33e − 03	8.86e − 04	1.43e − 01	1.43e − 01	−4.68e − 06
10	−3.73e − 03	−3.81e − 03	2.04e − 02	2.94e − 03	2.95e − 03	5.51e − 04	9.99e − 02	9.99e − 02	1.00e − 04

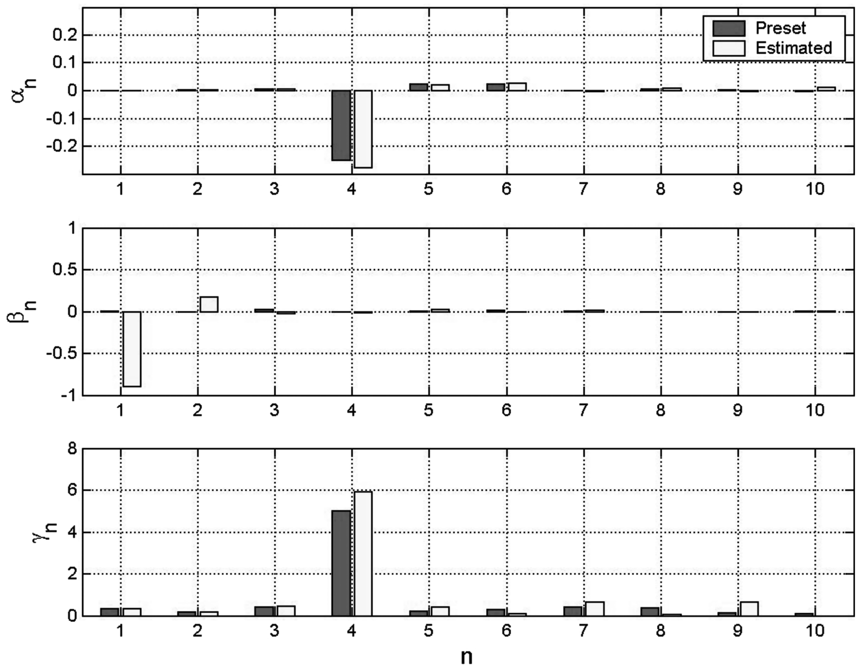

Fig. 5 Comparison of the preset and estimated correction coefficients α_n, β_n, and γ_n for the beam model when only translational DOFs of two modes are measured.

method based on the same consideration previously yields the result shown in Fig. 5. The estimations of the stiffness and damping terms, including those of the damaged element, are reasonably well, except the mass estimate of the first element. It is interesting to note that if two rotational DOFs associated with element 4 are also measured, then the result improves significantly as shown in Fig. 6.

Because mass terms usually can be modeled properly and are unlikely to change significantly even when damages occur, the remaining numerical study considers not updating mass terms. When only damping and stiffness terms are to be updated, using just one measured complex bending mode, along with 10 bending modes from the analytical model, is sufficient in the implementation of the CMCM method. Neglecting the update for the mass terms is equivalent to introducing the *modeling error* in mass because the unchanged masses of the adopted analytical model are different from those of the true model, which has been employed to generate measurements. Figure 7 compares the preset and estimated correction coefficients α_n and γ_n when only the 10 translational DOFs of the first mode are measured. The estimations for both damping and stiffness correction coefficients seem pretty good.

In practice, modal measurements always contain errors. Figure 8 shows the updating results based on using a corrupted measured mode that possesses a 0.1% proportional random error. Identical mass modeling error and spatial incompleteness mentioned in Fig. 7 are also included while obtaining Fig. 8. The value of the corrupted measured mode at each nodal point is generated by multiplying its true value with a factor $(1 + \epsilon)$, where ϵ has been simulated from a Gaussian random number generator with mean 0 and standard

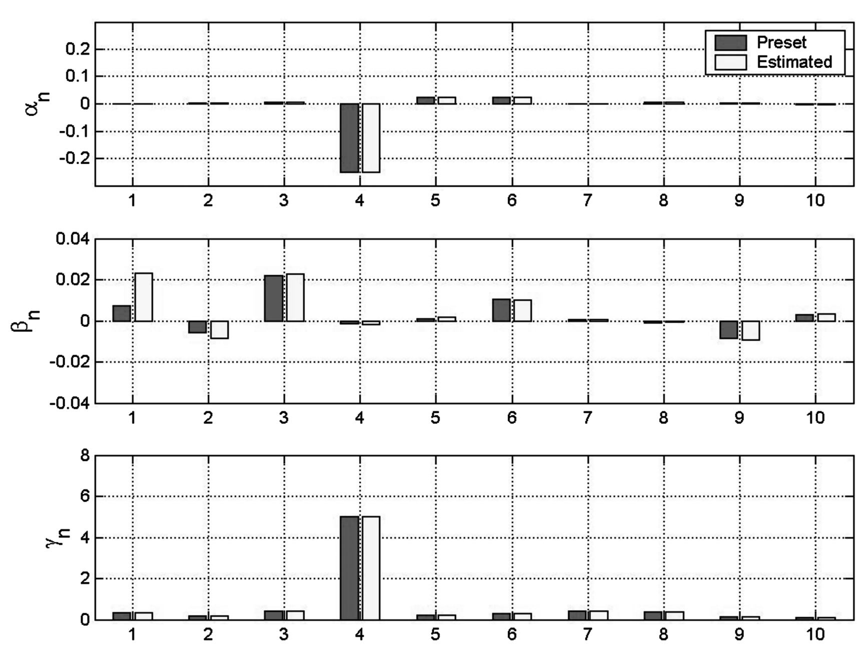

Fig. 6 Comparison of the preset and estimated correction coefficients α_n, β_n, and γ_n for the beam model when 10 translational and two rotational DOFs of two modes are measured.

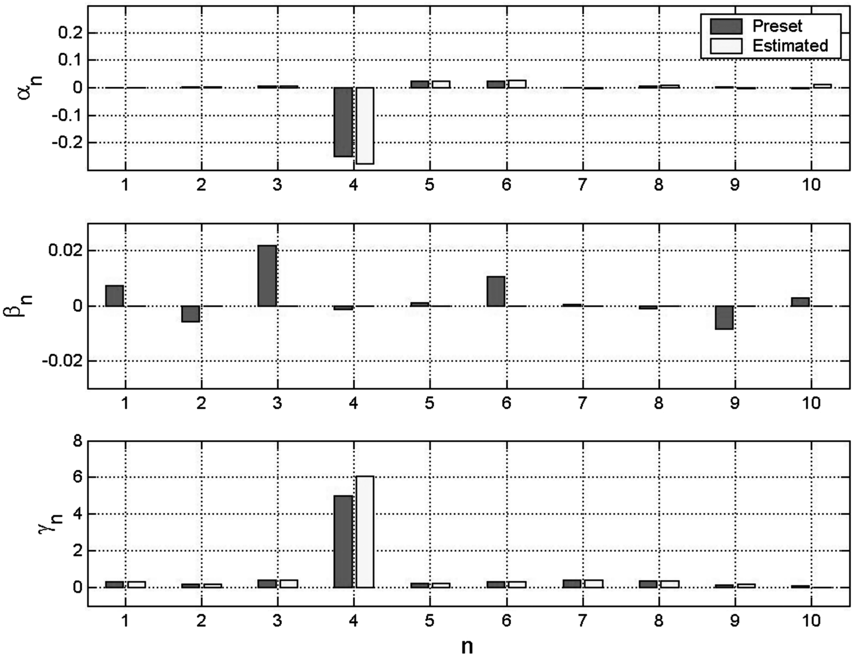

Fig. 7 Comparison of the preset and estimated correction coefficients α_n and γ_n for the beam model when β_n are not updated and 10 translational DOFs of the first mode only are measured.

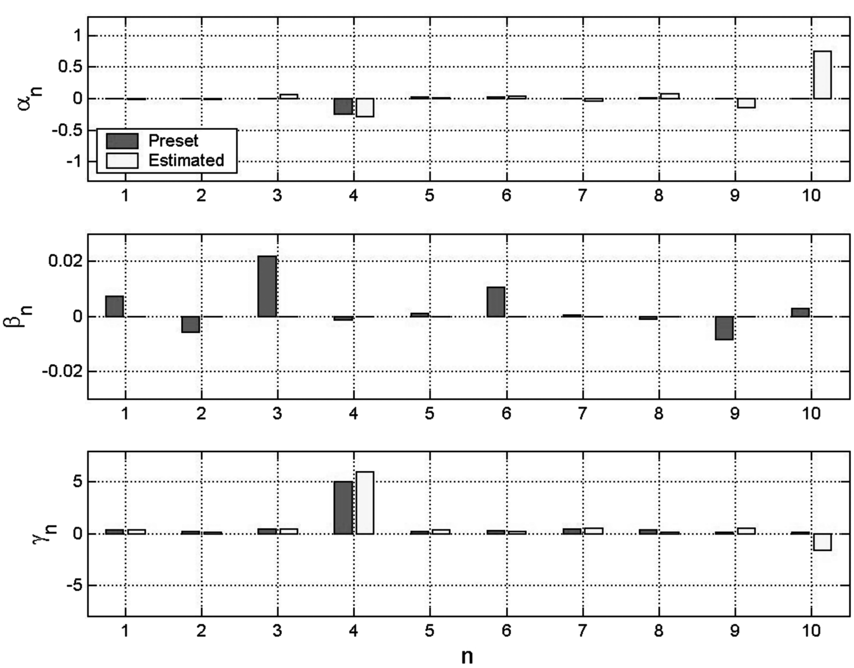

Fig. 8 Comparison of the preset and estimated correction coefficients α_n and γ_n for the beam model when β_n are not updated and 10 translational DOFs of a noisy second mode only are measured.

deviation 0.001. Figure 8 clearly exhibits the fact that the stiffness and damping terms at the free end of the beam are more influenced by the measurement noise.

Note that the numerical observations related to the mode incompleteness, modeling error, and measurement noise in this article are problem and parameterization dependent. A theoretical investigation of the effect on the CMCM method due to various sources of error remains to be done in the future.

Conclusions

This paper developed a simple, efficient, and systematic approach, named the cross-model cross-mode method, for solving model updating problems for linearly damped vibrating systems, where the damping matrix could be either proportional or nonproportional. The precise objective was to simultaneously update the mass, damping, and stiffness matrices of the finite element model for a dynamic system, when two or more "measured" complex modes were provided. The system reconstruction would satisfy modal and structural constraints simultaneously, where the *modal constraint* refers to matching the specified modes (eigenvalues and eigenvectors) and the *structural constraint* refers to keeping the desirable features associated with the mathematical model, including the physical connectivity of the finite element model, and the symmetric real-valued mass, damping, and stiffness matrices. The accuracy of the CMCM method was demonstrated numerically by two simulated examples, a 4-DOF mass–spring–damper system and a 30-DOF finite element model for a damaged cantilever beam. In both examples, when two spatially complete modes were available, the reconstruction of all system (mass, damping, and stiffness) matrices was found to be excellent. In the second example, when only 10 transverse DOFs associated with the 30-DOF mathematical model were measured, the reconstruction by the CMCM method, together with the classical Guyan reduction scheme, for all system matrices was found to be satisfactory also. The proposed CMCM method has provided a new direction on solving the quadratic inverse eigenvalue problems. The mathematical procedure of the CMCM method seems to have enough generality that, with some suitable modifications, it can be applied to other types of partially described inverse eigenvalue problems as well.

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